Tridiagonal Algorithms. Crout Factorization

A 4x4 example

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix}$$

Crout proposed a decomposition of A as $A = LU$, where

$$L = \begin{pmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ 0 & l_{32} & l_{33} & 0 \\ 0 & 0 & l_{43} & l_{44} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & u_{12} & 0 & 0 \\ 0 & 1 & u_{23} & 0 \\ 0 & 0 & 1 & u_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then,

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} l_{11} & l_{11}u_{12} & 0 & 0 \\ l_{21} & l_{21}u_{12}+l_{22} & l_{22}u_{23} & 0 \\ 0 & l_{32} & l_{32}u_{23}+l_{33} & l_{33}u_{34} \\ 0 & 0 & l_{43} & l_{43}u_{34}+l_{44} \end{pmatrix}$$

Therefore,

a) $l_{11} = a_{11}$

b) $l_{i,i-1} = a_{i,i-1}, \quad i = 2,..4$ (loop)

c) $l_{11} u_{12} = a_{12} \Rightarrow u_{12} = a_{12}/l_{11}$

d) $l_{21} u_{12} + l_{22} = a_{22} \Rightarrow l_{22} = a_{22} - l_{21} u_{12}$

e) $l_{22} u_{23} = a_{23} \Rightarrow u_{23} = a_{33}/l_{22}$

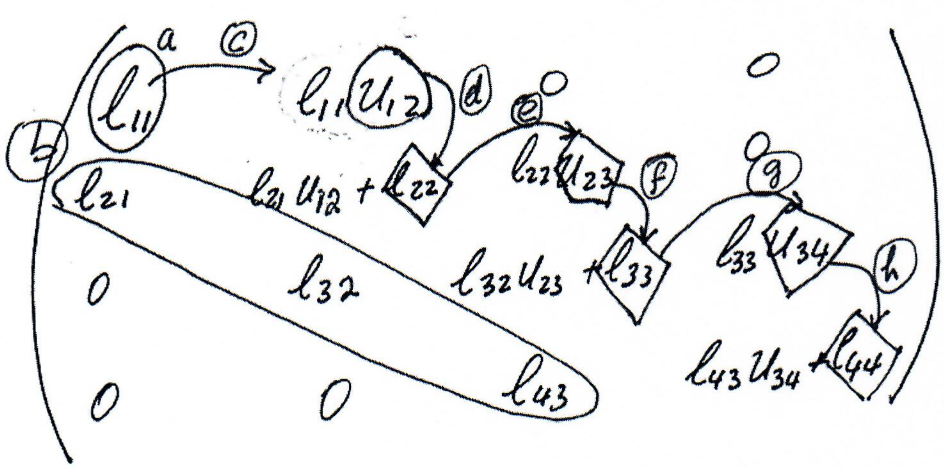
f) $l_{32} u_{23} + l_{33} = a_{33} \Rightarrow l_{33} = a_{33} - l_{32} u_{23}$

g) $l_{33} u_{34} = a_{34} \Rightarrow u_{34} = a_{34}/l_{33}$

h) $l_{43} u_{34} + l_{44} = a_{44} \Rightarrow l_{44} = a_{44} - l_{43} u_{34}$

$l_{i,i} = a_{i,i} - l_{i,i-1} u_{i-1,i}$

$u_{i,i+1} = a_{i,i+1}/l_{i,i}$



$$A = LU$$

Algorithm to find L and U:

$l_{11} = a_{11}$

For $i = 1 : n-1$

$\quad u_{i,i+1} = a_{i,i+1} / l_{ii}$ 1 M/D

$\quad l_{i+1,i} = a_{i+1,i}$

$\quad l_{i+1,i+1} = a_{i+1,i+1} - l_{i+1,i}\, u_{i,i+1}$ 1 M/D + 1 A/S.

end

Subtotal: M/D: $2(n-1)$

A/S: $n-1$

System Solution

$$A\vec{x}=\vec{b} \iff LU\vec{x}=\vec{b}$$

In two steps: I) $L\vec{y}=\vec{b}$

II) $U\vec{x}=\vec{y}$

(I)

$$\begin{pmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & l_{n,n-1} & l_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Forward Substitution:

$$l_{11}y_1 = b_1 \Rightarrow y_1 = b_1/l_{11} \quad \cdots\cdots \quad 1\ M/D$$

$$l_{21}y_1 + l_{22}y_2 = b_2 \Rightarrow y_2 = \frac{b_2 - l_{21}y_1}{l_{22}}$$

$$\vdots$$

$$l_{i,i-1}y_{i-1} + l_{i,i}y_i = b_i \Rightarrow y_i = \frac{b_i - l_{i,i-1}y_{i-1}}{l_{ii}} \quad \begin{matrix} 2\ M/D \\ 1\ A/S \end{matrix}$$

$i=2,\dots n$

Subtotal operations

M/D: $1 + 2(n-1) = 2n-1$

A/S: $n-1$.

(II) $U\vec{x} = \vec{y}$.

$$\begin{pmatrix} 1 & u_{12} & 0 & \cdots & 0 \\ 0 & 1 & u_{23} & \cdots & 0 \\ \vdots & & \ddots & \ddots & \\ \vdots & & & 1 & u_{n-1,n} \\ 0 & & & & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}$$

Backward Subst.

$x_n = y_n$

$x_{n-1} + u_{n-1,n} x_n = y_{n-1} \Rightarrow x_{n-1} = y_{n-1} - u_{n-1,n} x_n$

$\vdots$

$i = n-1, .. 1$

In general, $x_i = y_i - u_{i,i+1} x_{i+1}$

$i = n-1, .., 1$ $(n-1)$.

M/D: $\sum_{i=1}^{n-1} 1 = 1 \cdot (n-1) = n-1$

A/S: $\sum_{i=1}^{n-1} 1 = n-1$

Advantage:

$\mathcal{O}(n)$ operations

vs

$\mathcal{O}(n^3/3)$ operations in Gauss elimination.

TOTAL:

M/D: $2n-2 + 2n-1 + n-1 = \boxed{5n-4}$

A/S: $n-1 + n-1 + n-1 = \boxed{3n-3}$

Crout FACTORIZATION

It requires $l_{ii} \neq 0$, $i=1,2,..n$

True if A ① Diag. Dominant (strictly)

or

② Positive definite.

other

Thm 6.29

1) $A = [a_{ij}]$ is tridiag

2) $a_{i,i-1}, a_{i,i+1} \neq 0$, $i=2,3.., n-1$,

3) $|a_{11}| > |a_{12}|$,

4) $|a_{i,i}| \geq |a_{i,i+1}| + |a_{i,i-1}|$, $i = 2, 3.., n-1$

5) $|a_{nn}| > |a_{n,n-1}|$

then

A is nonsingular and the values of l_{ii} of the Crout Alg. are nonzero., $i=1,.,n$